

# Hilden Braid Groups

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## Abstract

Let  $H_g$  be a genus  $g$  handlebody and  $MCG_{2n}(T_g)$  be the group of the isotopy classes of orientation preserving homeomorphisms of  $T_g = \partial H_g$ , fixing a given set of  $2n$  points. In this paper we study two particular subgroups of  $MCG_{2n}(T_g)$  which generalize Hilden groups defined by Hilden in [16]. As well as Hilden groups are related to plate closures of braids, these generalizations are related to Heegaard splittings of manifolds and to bridge decompositions of links. Connections between these subgroups and motion groups of links in closed 3-manifolds are also provided.

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## 1 Introduction

In [16] Hilden introduced and found generators for two particular subgroups of the mapping class group of the sphere with  $2n$  punctures. Roughly speaking these groups consist of (the isotopy classes of) homeomorphisms of the punctured sphere which admit an extension to the 3-ball fixing  $n$  arcs embedded in the 3-ball and bounded by the punctures. The interest in these groups was motivated by the theory of links in  $\mathbb{S}^3$  (or in  $\mathbb{R}^3$ ). In [5], Hilden's generators were used in order to find a finite number of explicit equivalence moves relating two braids having the same plate closure. More recently, many authors investigated different groups related to Hilden's ones (see [1, 6, 26, 27]) and in particular motions groups (introduced in [12]). The second author introduced in [10] a higher genus generalization of Hilden's groups. These groups are subgroups of punctured mapping class groups of closed surfaces and are related to the study of link theory in a closed 3-manifold.

In this paper we define and study a different higher genus generalization of Hilden's groups: the Hilden braid groups. These groups can be seen as subgroups of the ones studied in [10] and they can be thought as a generalization of Hilden's groups in the "braid direction". Analogously to the genus zero case, our interest in these groups is mainly motivated by the theory of links in a closed 3-manifold. With

respect to groups introduced in [10], our groups seem to be more useful on studying links in a fixed manifolds.

The paper is organized as follows: Section 2 is devoted to the definition of Hilden braid groups and pure Hilden braid groups; a set of generators for these groups will be provided in Section 3 (Theorems 2 and 3). As we will prove in Section 4, by fixing a Heegaard decomposition of a given 3-manifold, it is possible to define a plat-like closure for  $2n$ -string braids on the Heegaard surface. In this setting, Hilden braid groups play a role similar to Hilden groups in the plat closure of classical braids. Moreover, as in the genus zero case, Hilden braid groups are connected with motion groups of links in closed 3-manifolds; this relation is established in Section 5 (Theorem 7).

## 2 Hilden groups: topological generalizations

Referring to Figure 2, let  $H_g$  be an oriented handlebody of genus  $g \geq 0$  and  $\partial H_g = T_g$ . We recall that a system of  $n$  pairwise disjoint arcs  $\mathcal{A}_n = \{A_1, \dots, A_n\}$  properly embedded in  $H_g$  is called *trivial* or *boundary parallel* if there exist  $n$  disks (the grey ones in Figure 1)  $D_1, \dots, D_n$ , called *trivializing disks*, embedded in  $H_g$  such that  $A_i \cap D_i = A_i \cap \partial D_i = A_i$ ,  $\partial D_i - A_i \subset \partial H_g$  and  $A_i \cap D_j = \emptyset$ , for  $i, j = 1, \dots, n$  and  $i \neq j$ .

By means of the trivializing disks  $D_i$  we can “project” each arc  $A_i$  into the arc  $a_i = \partial D_i - \text{int}(A_i)$  embedded in  $T_g$  and with the property that  $a_i \cap a_j = \emptyset$ , if  $i \neq j$ . We denote with  $P_{i,1}, P_{i,2}$  the endpoints of the arc  $A_i$  (which clearly coincide with the endpoints of  $a_i$ ), for  $i = 1, \dots, n$ .

Let  $\text{MCG}_{2n}(T_g)$  (resp.  $\text{MCG}_n(H_g)$ ) be the group of the isotopy classes of orientation preserving homeomorphisms of  $T_g$  (resp.  $H_g$ ) fixing the set  $\mathcal{P}_{2n} = \{P_{i,1}, P_{i,2} \mid i = 1, \dots, n\}$  (resp.  $A_1 \cup \dots \cup A_n$ ).

The *Hilden mapping class group*  $\mathcal{E}_{2n}^g$  is the subgroup of  $\text{MCG}_{2n}(T_g)$  defined as the image of the injective group homomorphism  $\text{MCG}_n(H_g) \longrightarrow \text{MCG}_{2n}(T_g)$  induced by restriction to the boundary. In other words,  $\mathcal{E}_{2n}^g$  consists of the isotopy classes of homeomorphisms that admit an extension to  $H_g$  fixing  $A_1 \cup \dots \cup A_n$ . Moreover, if  $\text{PMCG}_{2n}(T_g)$  denotes the subgroup of  $\text{MCG}_{2n}(T_g)$  consisting of the isotopy classes of the homeomorphisms of  $T_g$  fixing the punctures pointwise, we set  $\bar{\mathcal{E}}_{2n}^g = \text{PMCG}_{2n}(T_g) \cap \mathcal{E}_{2n}^g$  and call it the *pure Hilden mapping class group*. As recalled before, the groups  $\mathcal{E}_{2n}^0$  and  $\bar{\mathcal{E}}_{2n}^0$  were first introduced and studied by Hilden in [16], where the author found a finite set of generators, while in [10] has been provided a set of generators for all Hilden (pure) mapping class groups.

Now we are ready to define Hilden braid groups. Consider the commutative

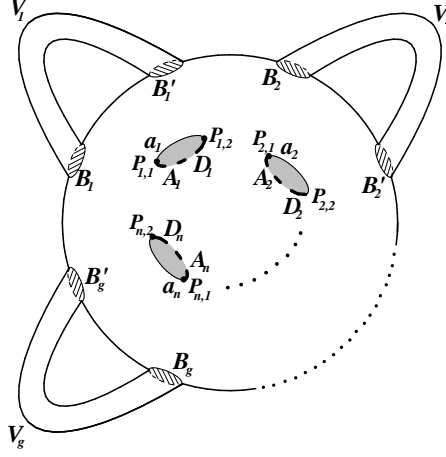


Figure 1: The model for a genus  $g$  handlebody and a trivial system of arcs.

diagram

$$\begin{array}{ccc}
 \text{MCG}_n(H_g) & \xrightarrow{\cong} & \mathcal{E}_{2n}^g \subset \text{MCG}_{2n}(T_g) \\
 \downarrow \bar{\Omega}_{g,n} & & \downarrow \Omega_{g,n} \\
 \text{MCG}(H_g) & \xrightarrow{\cong} & \mathcal{E}_0^g \subset \text{MCG}(T_g).
 \end{array}$$

where the vertical rows are forgetfull homomorphisms. The  $n$ -th *Hilden braid group* of the surface  $T_g$  is the group  $\text{Hil}_n^g := \mathcal{E}_{2n}^g \cap \ker \Omega_{g,n} \cong \ker \bar{\Omega}_{g,n}$ . The  $n$ -th *Hilden pure braid group*  $\text{PHil}_n^g$  of the surface  $T_g$  is the pure part of  $\text{Hil}_n^g$ , that is  $\text{Hil}_n^g \cap \text{PMCG}_{2n}(T_g)$ . Notice that, since  $\text{MCG}(\mathbb{S}^2) = \text{PMCG}(\mathbb{S}^2) = 1$ , then  $\mathcal{E}_{2n}^0 = \text{Hil}_n^0$  and  $\mathcal{E}_{2n}^0 = \text{PHil}_n^0$ . Moreover, in [3], it is shown that  $\ker(\Omega_{g,n})$  is isomorphic to the quotient of the braid group  $B_{2n}(T_g)$  by its center, which is trivial if  $g \geq 2$ . So, if  $g \geq 2$  we can see  $\text{Hil}_n^g$  as a subgroup of the braid group of the surface  $T_g$ .

In [26, 27], Tawn found a finite presentation for two groups that he called the Hilden group  $\mathbf{H}_{2n}$  and pure Hilden group  $\mathbf{PH}_{2n}$ . The definition proposed by Tawn is slightly different from ours; indeed,  $\mathbf{H}_{2n}$  and  $\mathbf{PH}_{2n}$  are subgroups of, respectively, the braid group  $B_{2n}$  and the pure braid group  $P_{2n}$ , instead of, respectively, the mapping class group  $\text{MCG}_{2n}(\mathbb{S}^2)$  and the pure mapping class group  $\text{PMCG}_{2n}(\mathbb{S}^2)$ . Nevertheless, from these presentations it is not difficult to obtain a presentation for  $\text{Hil}_n^0$  and  $\text{PHil}_n^0$  as sketched in the following. Consider the inclusion of the punctured  $2n$  disk into the  $2n$  punctured sphere: it induces surjective maps from  $B_{2n} = \text{MCG}_{2n}(D^2)$  to  $\text{MCG}_{2n}(\mathbb{S}^2)$  and from  $P_{2n} = \text{PMCG}_{2n}(D^2)$  to  $\text{PMCG}_{2n}(\mathbb{S}^2)$ . The kernels of these maps coincide with the subgroup normally generated by the center of  $\text{MCG}_{2n}(D^2)$  and the element  $\sigma_1 \dots \sigma_{2n-1}^2 \dots \sigma_1$ , where  $\sigma_i$  denotes the usual generator of the braid group  $B_{2n}$ . One can easily show that such elements belong to  $\mathbf{PH}_{2n}$  and that these maps restrict to surjective homomorphisms  $\mathbf{H}_{2n} \longrightarrow \text{Hil}_n^0$

and  $\mathbf{PH}_{2n} \longrightarrow P\text{Hil}_n^0$ . Therefore a finite presentation of  $\text{Hil}_n^0$  (respectively of  $P\text{Hil}_n^0$ ) is given by the same set of generators of  $\mathbf{H}_{2n}$  (respectively  $\mathbf{PH}_{2n}$ ) and the same set of relations of  $\mathbf{H}_{2n}$  (respectively  $\mathbf{PH}_{2n}$ ) plus the relation  $W_1 = 1$  and  $W_2 = 1$ , where  $W_1$  and  $W_2$  are, respectively, the generator of the center of  $\text{MCG}_{2n}(D^2)$  and the element  $\sigma_1 \dots \sigma_{2n-1}^2 \dots \sigma_1$  both written as words in the generators of  $\mathbf{H}_{2n}$  (respectively  $\mathbf{PH}_{2n}$ ). For further details on the genus zero case see [10], while in this paper we will mainly focus on the positive genus cases.

### 3 Generators of $\text{Hil}_n^g$

In this section we find a set of generators for  $\text{Hil}_n^g$  and  $P\text{Hil}_n^g$ . We start by fixing some notations.

Referring to Figure 2, for each  $k = 1, \dots, g$ , we denote with  $V_k$  the  $k$ -th 1-handle (i.e. a solid cylinder) obtained by cutting  $H_g$  along the two (isotopic) meridian disks  $B_k$  and  $B'_k$ . Moreover, we set  $b_k = \partial B_k$  and  $b'_k = \partial B'_k$  and call them meridian curves. For each  $i = 1, \dots, n$  the disk  $D_i$  denotes the trivializing disk for the  $i$ -th arc  $A_i$  and  $a_i = \partial D_i \setminus \text{int}(A_i)$ . The endpoints of both  $a_i$  and  $A_i$  are denoted with  $P_{i,1}, P_{i,2}$  and we set  $\mathcal{P}_{2n} = \{P_{i,1}, P_{i,2} \mid i = 1, \dots, n\}$ . We denote with  $D$  a disk embedded in  $T_g$  containing all the arcs  $a_i$  and not intersecting any meridian curve  $b_k$  or  $b'_k$ . Finally  $\delta_i$  denotes a disk in  $D$  containing  $a_i$  and such that  $\delta_i \cap a_j = \emptyset$  for  $i \neq j$  and  $j = 1, \dots, n$ .

Let us describe certain families of homeomorphisms of  $T_g$  fixing setwise  $\mathcal{P}_{2n}$  and whose isotopy classes belong to  $\text{Hil}_n^g$ . We will keep the same notation for a homeomorphism and its isotopy class.

**Intervals** For  $i = 1, \dots, n$ , we denote with  $\iota_i$  the homeomorphism of  $T_g$  that exchanges the endpoints of  $a_i$  inside  $\delta_i$  and that is the identity outside  $\delta_i$ . The interval  $\iota_i$  is also called braid twist (see for instance [21]).

**Elementary exchanges of arcs** For  $i = 1, \dots, n-1$ , let  $N_i$  be a tubular neighborhood of  $\delta_i \cup \beta_i \cup \delta_{i+1}$  where  $\beta_i$  is a band connecting  $\delta_i$  and  $\delta_{i+1}$ , lying inside  $D$  and not intersecting any arc  $a_j$ , for  $j = 1, \dots, n$ . We denote with  $\lambda_i$  the homeomorphism of  $T_g$  that exchanges  $a_i$  and  $a_{i+1}$ , mapping  $P_{i,j}$  to  $P_{i+1,j}$  inside  $N_i$ , for  $j = 1, 2$ , and that is the identity outside  $N_i$ .

**Elementary twists** We denote with  $s_i$  the Dehn twist along the curve  $d_i = \partial \delta_i$ . Notice that  $s_i = \iota_i^2$  in  $\text{MCG}_{2n}(T_g)$ .

**Slides of arcs** Let  $C$  be an oriented simple closed curve in  $T_g \setminus \mathcal{P}_{2n}$  intersecting  $a_i$  transversally in one point. Consider an embedded closed annulus  $A(C)$  in  $T_g$  whose core is  $C$ , containing  $a_i$  in its interior part and such that

$A(C) \cap \mathcal{P}_{2n} = \{P_{i,1}, P_{i,2}\}$ . We denote with  $C_1$  and  $C_2$  the boundary curves of  $A(C)$  with the convention that  $C_1$  is the one on the left of  $C$  according to its orientation, (see Figure 3). The slide  $S_{i,C}$  of the arc  $a_i$  along the curve  $C$  is the multi-twist  $T_{C_1}^{-1}T_{C_2}s_i^\varepsilon$ , where  $\varepsilon = 1$  if travelling along  $C$  we see  $P_{i,1}$  on the right and  $\varepsilon = -1$  otherwise. Such an element fixes  $a_i$  and determines on  $T_g$  the same deformation caused by “sliding” the arc  $a_i$  along the curve  $C$  according to its orientation. We denote the set of all the arc slides by  $\mathcal{S}_n^g$ .

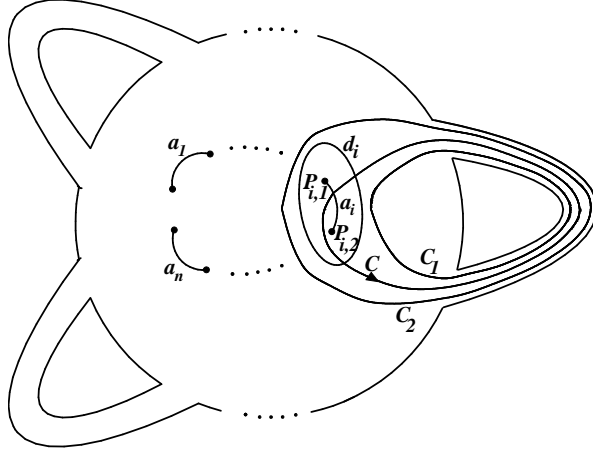


Figure 2: The slide  $S_{i,C} = T_{C_1}^{-1}T_{C_2}s_i$  of the arc  $a_i$  along the curve  $C$ .

**Admissible slides of meridian disks** Let  $T_g(i)$  be the genus  $g - 1$  surface obtained by cutting out from  $T_g$  the boundary of the  $i$ -th handle, and capping the resulting holes with the two meridian disks  $B_i$  and  $B'_i$  as in Figure 3. A simple closed curve  $C$  on  $T_g(i)$  will be called an admissible curve for the meridian disk  $B_i$  if it does not intersect  $B'_i \cup \mathcal{P}_{2n}$ , it intersects  $B_i$  in a simple arc and is homotopic to the trivial loop in  $T_g(i) \setminus B'_i$  rel  $Q$  where  $Q$  is any point of  $B_i \cap C$ . By exchanging the roles of  $B_i$  and  $B'_i$  we obtain the definition of admissible curve for the meridian disk  $B'_i$ .

Let  $C$  be an admissible oriented curve for the meridian disk  $B_i$ . Let  $A(C)$  be an embedded closed annulus in  $T_g(i) \setminus (B'_i \cup \mathcal{P}_{2n})$  whose core is  $C$  and containing  $B_i$  in its interior part. We denote with  $C_1$  and  $C_2$  the boundary curves of  $A(C)$  with the convention that  $C_1$  is the one on the left of  $C$  according to its orientation, (see Figure 3). Notice that one between  $C_1$  and  $C_2$  is homotopic to  $b_i$  in  $T_g(i) \setminus B'_i$ , while the other is trivial in  $T_g(i) \setminus B'_i$ . An admissible slide  $M_{i,C}$  of the meridian disk  $B_i$  along the curve  $C$  is the multi-twist  $T_{C_1}^{-1}T_{C_2}T_{b_i}^\varepsilon$ , where  $\varepsilon = 1$  if  $C_1$  is homotopic to  $b_i$  and  $\varepsilon = -1$  otherwise. Since this homeomorphism fixes both the meridian disks  $B_i$  and  $B'_i$ , it could be extended,

via the identity on the boundary of  $i$ -th handle, to a homeomorphism of  $T_g$ , and determines on  $T_g$  the same deformation caused by “sliding” the disk  $B_i$  along the curve  $C$  according to its orientation. In an analogous way we define  $M'_{i,C}$  an admissible slide of the meridian disk  $B'_i$  along an admissible oriented curve  $C$  for  $B'_i$ . We denote the set of all meridian slides with  $\mathcal{M}_n^g$ .

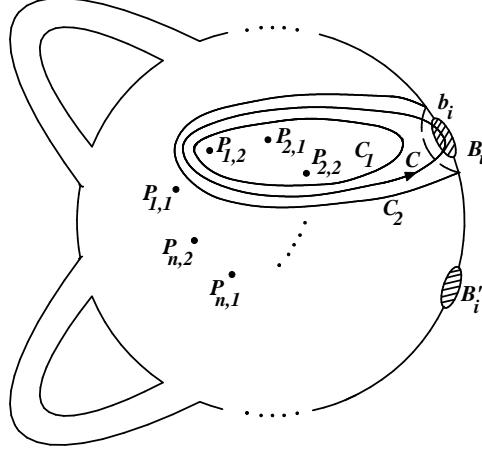


Figure 3: The slide  $M_{i,C} = T_{C_1}^{-1}T_{C_2}T_{b_i}^{-1}$  of the meridian disk  $B_i$  along the curve  $C$ .

REMARK 1 In [10] one can find explicit extensions of all above homeomorphisms to the couple  $(H_g, \mathcal{A}_n)$ , that is they all belongs to  $\mathcal{E}_{2n}^g$ . Moreover it is straightforward to see that all the above elements belong also to the kernel of  $\Omega_{g,n}$  and so to  $\text{Hil}_n^g$ .

It is possible to define the slide of a meridian disk  $B_i$  (resp.  $B'_i$ ) along a generic simple closed curve on  $T_g(i) \setminus B'_i \cup \mathcal{P}_{2n}$  (resp.  $T_g(i) \setminus B_i \cup \mathcal{P}_{2n}$ ). Such a meridian slide still belongs to  $\mathcal{E}_{2n}^g$ ; however, it is easy to see that a slide of a meridian disk belong to the kernel of  $\Omega_{g,n}$  (and so to  $\text{Hil}_n^g$ ) if and only if the sliding curve is admissible.

Let  $(Z_2)^n \rtimes S_n$  be the signed permutation group and let  $p : \text{MCG}_{2n}(T_g) \longrightarrow S_{2n}$  be the map which associates to any element of  $\text{MCG}_{2n}(T_g)$  the permutation induced on the punctures. The next proposition shows that  $\text{Hil}_n^g$  is generated by  $\iota_1$ ,  $\lambda_i$ , for  $i = 1, \dots, n-1$  and a set of generators for  $P\text{Hil}_n^g$ .

PROPOSITION 1 *The exact sequence*

$$1 \longrightarrow \text{PMCG}_{2n}(T_g) \longrightarrow \text{MCG}_{2n}(T_g) \longrightarrow S_{2n} \longrightarrow 1$$

*restricts to an exact sequence*

$$1 \longrightarrow P\text{Hil}_n^g \longrightarrow \text{Hil}_n^g \longrightarrow (Z_2)^n \rtimes S_n \longrightarrow 1.$$

*Proof.* The signed permutation group can be considered as the subgroup of  $S_{2n}$  generated by the transposition  $(1\ 2)$  and the permutations  $(2i-1\ 2i+1)(2i\ 2i+2)$ , for  $i = 1, \dots, n-1$ . Let  $\sigma \in \text{Hil}_n^g$ : since the extension of  $\sigma$  induces a permutation of the arcs  $A_1, \dots, A_n$ , then  $p(\sigma) \in (Z_2)^n \rtimes S_n$ . Moreover if  $p(\sigma) = 1$  then an extension of it fixes the arcs pointwise, so  $\sigma \in \text{PHil}_n^g$ .  $\blacksquare$

We say that an element  $\sigma \in \text{PHil}_n^g$  is an *arcs-stabilizer* if  $\sigma$  is the identity on  $a_i$ , for each  $i = 1, \dots, n$ . The set of all arcs-stabilizer elements of  $\text{PHil}_n^g$  determines a subgroup of  $\text{PHil}_n^g$  that we call the *arcs-stabilizer subgroup*.

The subgroup  $FP_n(T_g)$  of  $\ker(\Omega_{g,n}) \cap \text{PMCG}_{2n}(T_g)$ , consisting of the elements fixing the arcs  $a_1, \dots, a_n$  is called in [2] *n-th framed pure braid group of  $\Sigma_g$* : this group is a (non trivial) generalization of pure framed braid groups considered in [20, 23] and several equivalent definitions have been provided in [2].

**PROPOSITION 2** *Let  $FP_n(T_g)$  be the n-th framed braid group of  $T_g$  defined above. The arcs-stabilizer subgroup of  $\text{PHil}_{2n}^g$  coincides with  $FP_n(T_g)$ . In particular the arcs-stabilizer subgroup of  $\text{PHil}_{2n}^g$  is generated by the elementary twists  $s_i$  and the slides  $m_{i,j}$  and  $l_{i,j}$  of the arc  $a_i$  along the curves  $\mu_{i,j}$  and  $\lambda_{i,j}$  depicted in Figure 4, for  $i = 1, \dots, n$  and  $j = 1, \dots, g$ . As a consequence, the slide  $t_{i,k}$  of the arc  $a_i$  along the curve  $\tau_{i,k}$  depicted in Figure 5, is the composition of the above elementary twists and slides, for  $1 \leq i < k \leq n$ .*

*Proof.* By the above definitions, it is enough to show that  $FP_n(T_g)$  is a subgroup of  $\text{PHil}_n^g$ . In [2] is shown that  $FP_n(T_g)$  is generated by the elementary twists  $s_i$  and the slides  $m_{i,j}$  and  $l_{i,j}$  of the arc  $a_i$  along the curves  $\mu_{i,j}$  and  $\lambda_{i,j}$  depicted in Figure 4, for  $i = 1, \dots, n$  and  $j = 1, \dots, g$ . Therefore  $FP_n(T_g)$  is a subgroup of  $\text{PHil}_n^g$ . The multitwists  $t_{i,k}$ , for  $1 \leq i < k \leq n$ , belong to  $FP_n(T_g)$ : in particular they can be obtained by the above set of generators using lantern relations (see [2]).  $\blacksquare$

Now we will prove that an infinite set of generators for  $\text{PHil}_n^g$  is given by the elementary twists, all the arc slides and all the admissible meridian slides.

**THEOREM 2** *The group  $\text{PHil}_n^g$  is generated by  $\mathcal{M}_n^g \cup \mathcal{S}_n^g \cup \{s_1, \dots, s_n\}$ .*

*Proof.* Let  $\mathcal{G}_n^g$  be the subgroup of  $\text{PHil}_n^g$  generated by  $\mathcal{M}_n^g \cup \mathcal{S}_n^g$ . By Proposition 2 it is enough to show that for any  $\sigma \in \text{PHil}_n^g$  there exists an element  $h \in \mathcal{G}_n^g$  such that  $h\sigma$  is an arcs-stabilizer. In order to do so, the first step will be to find an element  $h_n \in \mathcal{G}_n^g$  such that

- (i)  $h_n\sigma(a_n) = a_n$ ;
- (ii) for each  $i = 1, \dots, n$  such that  $\sigma(a_n) \cap a_i = \emptyset$  we have  $h_n(a_i) = a_i$ .

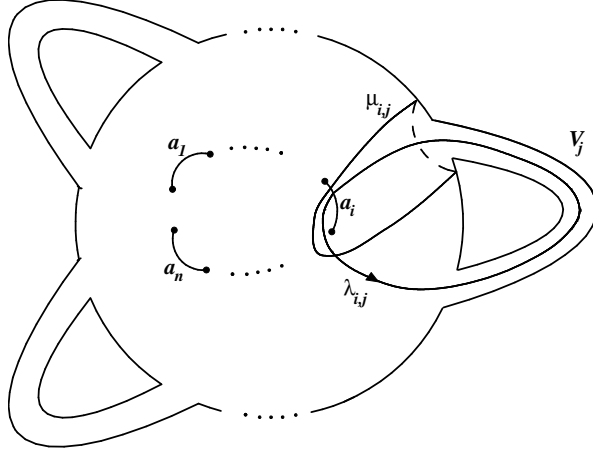


Figure 4: The arc slides  $m_{i,j}$  and  $l_{i,j}$ .

The element  $h_n$  will be defined as the composition of arc slides and admissible meridian slides along opportunely chosen curves.

We denote with  $\mathcal{D}$  the union of all the disk  $D_i$  for  $i = 1, \dots, n$  and let  $\mathcal{I} = \sigma(D_n) \cap \mathcal{D}$ . Up to isotopy, we can assume that  $\mathcal{I}$  consists of a finite number of arcs. Clearly, each arc  $l$  in  $\mathcal{I}$  is a component of  $\sigma(D_n) \cap D_k$  for a unique  $k$ ; moreover, since  $\sigma(A_n) \cap A_k = A_n \cap A_k$ , if  $k \neq n$  then  $\sigma(A_n) \cap A_k = \emptyset$ , so the endpoints of  $l$  belong to  $\sigma(a_n) \cap a_k$ . If, instead,  $k = n$  we can assume that at least one endpoint of  $l$  belongs to  $\sigma(a_n) \cap a_n$ , since if both the endpoints lie in  $A_n$  then, by composing with a homeomorphism isotopic to the identity, the intersection  $l$  can be removed.

By an innermost argument, it is possible to choose  $l_0 \in \mathcal{I}$  with  $l_0 \subset \sigma(D_n) \cap D_k$  such that  $l_0$  determines a disk both in  $\sigma(D_n)$  and in  $D_k$ , whose union is a disk  $\bar{D}$ , properly embedded in  $H_g$ . Moreover  $\bar{D}$  intersects  $\sigma(D_n) \cap \mathcal{D}$  only in  $l_0$  if  $k \neq n$ , while, if  $k = n$  and one of the endpoints of  $l_0$  lies in  $A_n$ , then the intersection of  $\bar{D}$  with  $\sigma(D_n) \cap \mathcal{D}$  is an arc which is the union of  $l_0$  with a subarc of  $A_n$  going from  $l_0 \cap A_n$  to one of the punctures  $\partial A_n = \{P_{n,1}, P_{n,2}\}$ . In any case, the boundary of  $\bar{D}$  is the union of two simple arcs  $m_1$  and  $m_2$  on  $H_g$  with  $m_1 \subset \sigma(a_n)$  and  $m_2 \subset a_k$  (see Figure 6).

We set  $K_1 = \{k \mid \bar{D} \cap V_k \neq \emptyset\}$  and  $K_2 = \{1, \dots, g\} \setminus K_1$ . Notice that if  $k \in K_1$ , then for each arc  $\alpha \in \bar{D} \cap B_k$ , there exists a corresponding arc  $\alpha' \in \bar{D} \cap B'_k$  such that the union of the disks bounded by  $\alpha$  and  $\alpha'$  on  $B_k$ ,  $B'_k$  and  $\bar{D}$  is a properly embedded disk in the handle  $V_k$  bounding a ball (see Figure 6).

Indeed, if this is not the case, the intersection  $\alpha$  could be removed composing with an element isotopic to the identity. If  $H_g(K_2)$  denotes the handlebody (of genus  $g - |(K_2)|$ ) obtained from  $H_g$  by removing all the handles  $V_k$  with  $k \in K_2$ , then  $\bar{D}$  separates  $H_g(K_2)$  into two connected components  $\Delta_1$  and  $\Delta_2$ . Let  $\Delta$  be the



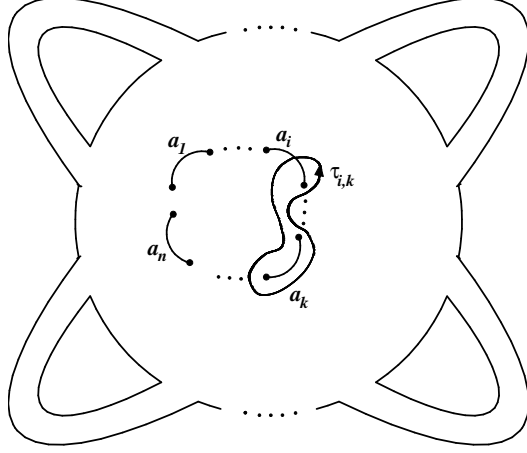


Figure 5: The arc slide  $t_{i,k}$ .

connected component of  $H_g(K_2) \setminus \bar{D}$  that does not contain  $A_k$ , which is the dotted zone in Figure 3.

Each disk  $D_i$  with  $i \neq k$  and each meridian disk  $B_k$  or  $B'_k$ , with  $k \in K_2$  is either contained or disjoint from  $\Delta$ . Let  $I_1 = \{i \mid A_i \subset \Delta\}$ ,  $I_2 = \{i \mid i \in K_2, B_i \subset \Delta\}$  and  $I_3 = \{i \mid i \in K_2, B'_i \subset \Delta\}$ . For each  $i \in I_1 \cup I_2 \cup I_3$  we choose a simple closed oriented curve  $C_i$  on  $H_g(K_2)$  such that

(\*) both  $C_i \cap h(a_n) = C_i \cap m_1$  and  $C_i \cap a_k = C_i \cap m_2$  consist of a single point; travelling along  $C_i$  the intersection with  $a_k$  comes before the one with  $\sigma(a_n)$ ; if  $i \in I_1$  then  $C_i$  intersects  $a_i$  in a single point, while if  $i \in I_2$  (resp.  $I_3$ ) then  $C_i$  is an admissible curve for the meridian disk  $B_i$  (resp.  $B'_i$ );  $C_i$  does not intersect all the others arcs  $a_j$  and all the other meridian disks  $B_j, B'_j$ , with  $j \in K_2$ .

To see that such a  $C_i$  exists, consider an arc  $a$  that starts from  $a_i$ ,  $B_i$  or  $B'_i$  travels inside  $\partial\Delta$  to  $m_1$  without intersecting all the other arcs  $a_j$  and all the other meridian disks  $B_j, B'_j$ , goes along  $m_1$  to  $a_k$  and finally goes along  $a_k$  to an endpoint of  $a_k$  (that is  $P_{k,1}$  or  $P_{k,2}$ ) without travelling along  $m_2$ . Then  $C_i$  can be chosen as the boundary of a small tubular neighborhood of  $a$ . In Figure 3 it is depicted the case  $I_1 = \{i\}$ ,  $I_2 = \{i'\}$  and  $I_3 = \emptyset$ .

If we set  $f_0 = \prod_{i \in I_1} S_{i,C_i} \prod_{i \in I_2} M_{i,C_i} \prod_{i \in I_3} M'_{i,C_i}$  then  $f_0(\Delta)$  does not contain any arc and any meridian disk. This means that  $f_0(\Delta)$  bounds a ball in  $H_g \setminus \mathcal{A}_n$  and so, up to composing with an element isotopic to the identity, we can remove the intersection  $l_0$ . Moreover,  $f_0\sigma(D_n) \cap \mathcal{D} = \mathcal{I} \setminus l_0$ .

By repeating the above procedure a finite number of times, we get an element  $f = f_k f_{k-1} \cdots f_0 \in \mathcal{G}_n^g$  such that  $f\sigma(D_n) \cap \mathcal{D} = A_n$ . Then  $f\sigma(D_n) \cup D_n$  is a

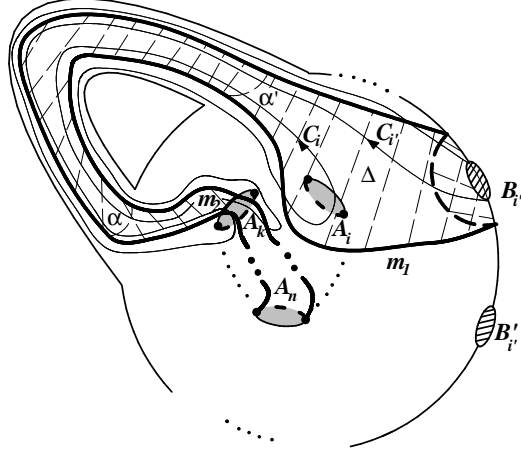


Figure 6:

properly embedded disk  $\tilde{D}$  in  $H_g$ . If we denote with  $H_g(K_2)$  the handlebody obtained from  $H_g$  by removing all the handles that have no intersection with  $\tilde{D}$  (that is  $K_2 = \{k \mid V_k \cap \tilde{D} = \emptyset\}$ ), then  $\tilde{D}$  separates  $H_g(K_2)$  into two connected components  $\Delta_1$  and  $\Delta_2$ . As above, each disk  $D_i$  with  $i \neq n$  and each meridian disk  $B_i$  or  $B'_i$ , corresponding to the removed handles, is either contained or disjoint from  $\Delta_k$ , for  $k = 1, 2$ . If there exists  $k = 1, 2$  such that  $\Delta_k$  does not contain any disk  $D_i, B_i$  or  $B'_i$  then, up to composing with an element isotopic to the identity, we have  $f\sigma(D_n) = D_n$  and so  $h_n = f$ . If this is not the case, we choose one of the two connected components, for example  $\Delta_1$ , and, as before, we set  $I_1 = \{i \mid i \in K_2, A_i \subset \Delta_1\}$ ,  $I_2 = \{i \mid i \in K_2, B_i \subset \Delta_1\}$  and  $I_3 = \{i \mid i \in K_2, B'_i \subset \Delta_1\}$  and for each  $i \in I_1 \cup I_2 \cup I_3$  we choose a simple oriented closed curve  $C_i$  on  $H_g(K_2)$  satisfying (\*). Then by taking  $h_n = \prod_{i \in I_1} S_{i, C_i} \prod_{i \in I_2} M_{i, C_i} \prod_{i \in I_3} M'_{i, C_i} f$  we get  $h_n \sigma(a_n) = a_n$ . Moreover  $h_n$  satisfies (ii) since it is the compositions of slides (of arcs or of meridian disks) along curves that by (\*) intersects only the arcs  $a_i$  such that  $a_i \cap \sigma(a_n) \neq \emptyset$ , and so fixes all the other arcs.

We can repeat the same procedure on  $a_{n-1}$ , that is we can find  $h_{n-1} \in \mathcal{G}_n^g$  with  $h_{n-1} h_n \sigma(a_{n-1}) = a_{n-1}$  and such that for each  $i = 1, \dots, n$  such that  $h_n \sigma(a_{n-1}) \cap a_i = \emptyset$  we have  $h_{n-1}(a_i) = a_i$ . This implies that  $h_{n-1} h_n \sigma(a_n) = a_n$ , that is  $h_{n-1} h_n \sigma$  fixes the last two arcs. Proceeding in this way, we construct, for each  $i = 1, \dots, n$ , an element  $h_i \in \mathcal{G}_n^g$  such that  $h_i h_{i+1} \dots h_n \sigma(a_j) = a_j$  for each  $j \geq i$ . So  $h = h_1 h_2 \dots h_{n-1} h_n$  is the required element of  $\mathcal{G}_n^g$ , that is  $h\sigma$  is arc-stabilizer. ■

In order to find a finite set of generators it would be enough to show that the subgroup of  $P\text{Hil}_n^g$  generated by  $\mathcal{S}_n^g$  and the one generated by the  $\mathcal{M}_n^g$  are finitely generated. The next two propositions show that the first subgroup is finitely generated, and the second one is finitely generated when  $g = 1$ .

PROPOSITION 3 *The subgroup of  $\text{PHil}_n(\text{T}_g)$  generated by  $\mathcal{S}_n^g$  is finitely generated by*

- (1) *the elementary twist  $s_i$ , for  $i = 1, \dots, n$ ;*
- (2) *the slides  $m_{i,j}$  and  $l_{i,j}$  of the arc  $a_i$  along the curves  $\mu_{i,j}$  and  $\lambda_{i,j}$  depicted in Figure 4, for  $i = 1, \dots, n$  and  $j = 1, \dots, g$ .*
- (3) *the slides  $s_{i,k}$ , of the arc  $a_i$  along the curve  $\sigma_{i,k}$  depicted in Figure 7, for  $1 \leq i \neq k \leq n$ .*

*Proof.* By the definition of slide if  $C \simeq C_1 \cdots C_r$  in  $\pi_1(\text{T}_g \setminus \{P_{j,k} \mid j = 1, \dots, n, j \neq i, k = 1, 2\}, *)$  then  $S_{i,C} = S_{i,C_r} \cdots S_{i,C_1}$  in  $\text{MCG}_{2n}(\text{T}_g)$ , where  $*$   $\in a_i$  is any fixed point. Then all the slides of the  $i$ -th arc are generated by the slides of the  $i$ -th arc along a set of generators for  $\pi_1(\text{T}_g \setminus \{P_{j,k} \mid j = 1, \dots, n, j \neq i, k = 1, 2\}, *)$ . The set of slides  $m_{i,j}$ ,  $l_{i,j}$  for  $j = 1, \dots, g$  and  $s_{i,k}$ ,  $s'_{i,k}$ , for  $k = 1, \dots, n$ ,  $i \neq k$  is therefore a set of generators for all the slides of the  $i$ -th arc, where  $s'_{i,k}$  is the slide of the arc  $a_i$  along the curve  $\sigma'_{i,k}$  depicted in Figure 7. Applying a lantern relation one obtains that  $s_{i,k}s'_{i,k} = t_{i,k}s_i^{-1}$ . The statement therefore, follows from the fact that, by Proposition 2, the slide  $t_{i,k}$  can be written as composition of the slides  $m_{j,r}$ ,  $l_{j,r}$  and elementary twists.  $\blacksquare$

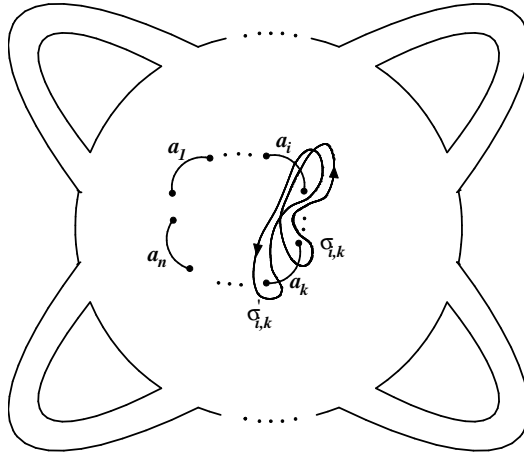


Figure 7: The arc slides  $s_{i,k}$  and  $s'_{i,k}$ .

PROPOSITION 4 *The subgroup of  $\text{PHil}_n(\text{T}_1)$  generated by  $\mathcal{M}_n^1$  is finitely generated.*

*Proof.* Let  $\mathbb{S}^2$  be the sphere obtained by cutting out from  $\text{T}_1$  the boundary of the handle, and capping the resulting holes with the two meridian disks  $B_1$  and  $B'_1$  as in the definition of meridian slides. Any simple closed curve  $C$  on  $\mathbb{S}^2$  which does

not intersect  $B'_1 \cup \mathcal{P}_{2n}$  and intersects  $B_1$  in a simple arc, is an admissible curve for the meridian disk  $B_1$ . Therefore, if we write  $C$  as a product  $C_1 \cdots C_r$  of a finite set of generators of  $\pi_1(\mathbb{S}^2 \setminus (\mathcal{P} \cup B'_1), *)$ , where  $*$   $\in B_1$  is any fixed point, we have  $M_{i,C} = M_{i,C_r} \cdots M_{i,C_1}$ . An analogous statement holds for  $M'_{i,C}$ .  $\blacksquare$

From these Propositions it follows that  $\text{PHil}_n^1$  is finitely generated. The problem of whether  $\text{PHil}_n^g$  is finitely generated or not while  $g > 1$  remains open. We end this section by giving the generators for  $\text{Hil}_n^g$ .

**THEOREM 3** *The group  $\text{Hil}_n^g$  is generated by*

- (1)  $\iota_1$  and  $\lambda_j$ ,  $j = 1, \dots, n-1$ ;
- (2)  $m_{1,k}$ ,  $l_{1,k}$ ,  $s_{1,r}$  with  $k = 1, \dots, g$  and  $r = 1, \dots, n$ ;
- (3) the elements of  $\mathcal{M}_n^g$ .

*Proof.* The statement follows from Theorem 2, Proposition 3 and the remark that  $s_1 = \iota_1^2$  and that arc slides of the arc  $a_i$  can be reduced to arc slides of the arc  $a_1$  by using (compositions of) elementary exchanges of arcs.  $\blacksquare$

## 4 Generalized plat closure

One of the main motivations to study topological generalizations of Hilden groups comes from link theory in 3-manifolds. In this section we describe a representation of links in 3-manifolds via braids on closed surfaces: this approach generalizes the concept of plat closure and explains the role played by  $\text{Hil}_n^g$  in this representation. We start by recalling the definition of  $(g, n)$ -links.

Let  $L$  be a link in a 3-manifold  $M$ . We say that  $L$  is a  $(g, n)$ -link if there exists a genus  $g$  Heegaard surface  $S$  for  $M$  such that

- (i)  $L$  intersects  $S$  transversally and
- (ii) the intersection of  $L$  with both of handlebodies into which  $M$  is divided by  $S$ , is a trivial system of  $n$  arcs.

Such a decomposition for  $L$  is called  $(g, n)$ -decomposition or  $n$ -bridge decomposition of genus  $g$ . The minimum  $n$  such that  $L$  admits a  $(g, n)$ -decomposition is called *genus  $g$  bridge number* of  $L$ .

Clearly if  $g = 0$  we get the usual notion of bridge decomposition and bridge number of links in the 3-sphere (or in  $\mathbb{R}^3$ ). Given two links  $L \subset M$  and  $L' \subset M'$  we say that  $L$  and  $L'$  are *equivalent* if there exists an orientation preserving homeomorphism  $f : M \longrightarrow M'$  such that  $f(L) = L'$  and we write  $L \cong L'$ .

The notion of  $(g, n)$ -decompositions was used in [10] to develop an algebraic representation of  $\mathcal{L}_{g,n}$ , the set of equivalence classes of  $(g, b)$ -links, as follows. Let  $(H_g, \mathcal{A}_n)$  be as in Figure 1 and let  $(\bar{H}_g, \bar{\mathcal{A}}_n)$  be a homeomorphic copy of  $(H_g, \mathcal{A}_n)$ . Fix an orientation reversing homeomorphism  $\tau : H_g \longrightarrow \bar{H}_g$  such that  $\tau(A_i) = \bar{A}_i$ , for each  $i = 1, \dots, n$ . Then the following application is well defined and surjective

$$\Theta_{g,n} : \text{MCG}_{2n}(\text{T}_g) \longrightarrow \mathcal{L}_{g,n} \quad \Theta_{g,n}(\psi) = L_\psi \quad (1)$$

where  $L_\psi$  is the  $(g, n)$ -link in the 3-manifold  $M_\psi$  defined by

$$(M_\psi, L_\psi) = (H_g, \mathcal{A}_n) \cup_{\tau\psi} (\bar{H}_g, \bar{\mathcal{A}}_n).$$

This means that it is possible to describe each link admitting a  $(g, n)$ -decomposition in a certain 3-manifold by a element of  $\text{MCG}_{2n}(\text{T}_g)$ . This element is not unique, since we have the following result.

**PROPOSITION 5** ([10]) *If  $\psi$  and  $\psi'$  belong to the same double coset of  $\mathcal{E}_{2n}^g$  in  $\text{MCG}_{2n}(\text{T}_g)$  then  $L_\psi \cong L_{\psi'}$ .*

Therefore, in order to describe all  $(g, n)$ -links via (1) it is enough to consider double coset classes of  $\mathcal{E}_{2n}^g$  in  $\text{MCG}_{2n}(\text{T}_g)$ . This representation has revealed to be a useful tool for studying links in 3-manifolds, see [7, 8, 11, 19]. However, if we represent links using (1), we have to deal with links that lie in different manifolds. If we want to fix the ambient manifold, then the following remark holds.

**REMARK 4** If  $\psi_1, \psi_2 \in \text{MCG}_{2n}(\text{T}_g)$  are such that  $\Omega_{g,n}(\psi_1) = \Omega_{g,n}(\psi_2)$  then  $L_{\psi_1}$  and  $L_{\psi_2}$  belong to the same ambient manifold.

So, in order to fix the ambient manifold, we want to modify representation (1) by separating the part that determines the manifold from the part that determines the link.

Referring to Figure 1, let  $D$  be a disk embedded in  $\text{T}_g$  containing all the arcs  $a_i$  and not intersecting any meridian curve  $b_k$  or  $b'_k$ , for  $i = 1, \dots, n$  and  $k = 1, \dots, g$ . Let  $\mathcal{T}_n^g$  be the subgroup of  $\text{MCG}_{2n}(\text{T}_g)$  generated by Dehn twist along curves that do not intersect the disk  $D$ . We have the following proposition.

**PROPOSITION 6** *For each  $\psi \in \mathcal{T}_n^g$  the link  $L_\psi$  is a  $n$  component trivial link in  $M_\psi$ .*

*Proof.* Since, the action of  $\psi$  on the punctures is trivial, for each  $i = 1, \dots, n$ , the arc  $A_i$  is glued, via  $\tau\psi$ , to  $\bar{A}_i$ , giving rise to a connected component of  $L_\psi$ . Moreover for each  $i = 1, \dots, n$  we have  $\psi(a_i) = a_i$ , so if we set  $\tau(D_i) = \bar{D}_i \subset \bar{H}_g$ , the  $i$ -th component  $A_i \cup_{\tau\psi} \bar{A}_i$  of  $L_\psi$  bounds in  $M_\psi$  the embedded disk  $D_i \cup_{\tau\psi} \bar{D}_i$ . ■

Let  $\text{T}_{g,1}$  be the compact surface obtained by removing the interior part of the disk  $D$ . The natural inclusion of  $\text{T}_{g,1}$  into  $\text{T}_g$  with  $2n$  marked points induces an

injective map  $\text{MCG}(T_{g,1}) \longrightarrow \text{MCG}_{2n}(T_g)$  (see [25]) and the mapping class group  $\text{MCG}(T_{g,1})$  turns out to be isomorphic to the group  $\mathcal{T}_n^g$ . On the other hand, we have also the following exact sequence:

$$1 \longrightarrow \pi_1(UT_{g,1}) \longrightarrow \mathcal{T}_n^g \longrightarrow \text{MCG}(T_g) \longrightarrow 1$$

where  $UT_{g,1}$  is the unit tangent bundle of  $T_{g,1}$  (see [2]). As a consequence each element of  $\text{MCG}(T_g)$  admits a lifting as an element of  $\mathcal{T}_n^g$ , so we can realize any genus  $g$  Heegaard decomposition of a 3-manifold  $M$  using an element of  $\mathcal{T}_n^g$ , for any  $n > 0$ . Now we are ready to define the *generalized plat closure*. Let  $M$  be a fixed manifold, and choose an element  $\psi \in \mathcal{T}_n^g$  such that  $M = M_\psi$ . We define a map

$$\Theta_{g,n}^\psi : \ker(\Omega_{g,n}) \longrightarrow \{(g, n) - \text{links in } M_\psi\} \quad (2)$$

given by  $\Theta_{g,n}^\psi(\sigma) = \Theta_{g,n}(\psi\sigma)$ . We set  $\hat{\sigma}^\psi = \Theta_{g,n}^\psi(\sigma)$ .

REMARK 5 As recalled before,  $\ker(\Omega_{g,n})$  is isomorphic to the braid group  $B_{2n}(T_g)$ , quotiented by its center, which is trivial if  $g \geq 2$ . This means that we can interpretate  $\Theta_{g,n}^\psi$  as a generalization of the notion of plat closure for classical braids, as shown schematically in Figure 8. Indeed, for  $g = 0$  and  $\psi = \text{id}$  we obtain the classical plat closure. This is the only representation in the case of the 3-sphere (i. e. with  $g = 0$ ), since  $\mathcal{T}_n^0$  is trivial. On the contrary, the generalized plat closure in a 3-manifold different from  $S^3$  depends on the choice of the element  $\psi \in \mathcal{T}_n^g$ , and, topologically, this corresponds to the choice of a Heegaard surface of genus  $g$  for  $M$ .

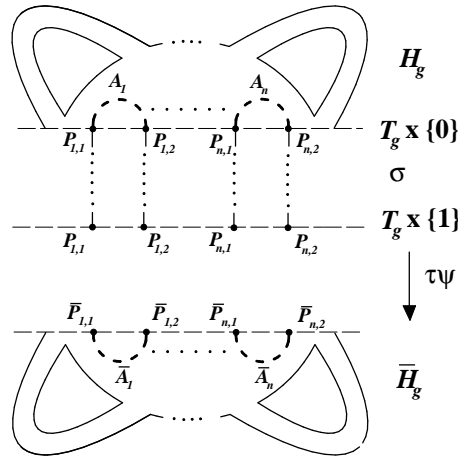


Figure 8: A generalized plat closure

In this setting a natural question arises.

QUESTION 1 *Is it possible to determine when two element  $\sigma_1 \in \ker(\Omega_{g,n_1})$  and  $\sigma_2 \in \ker(\Omega_{g,n_2})$  determine equivalent links via (2)?*

A partial answer is given by the following statement, that is a straightforward corollary of Proposition 5.

COROLLARY 6 *Let  $\psi \in \mathcal{T}_n^g$ . Denote with  $\text{Hil}_n^g(\psi) = \psi^{-1}\text{Hil}_n^g\psi$ .*

- 1) *if  $\sigma_1$  and  $\sigma_2$  belong to the same left coset of  $\text{Hil}_n^g$  in  $\ker(\Omega_{g,n})$  then  $\hat{\sigma}_1^\psi$  and  $\hat{\sigma}_2^\psi$  are equivalent links in the manifold  $M_\psi$ .*
- 2) *if  $\sigma_1$  and  $\sigma_2$  belong to the same right coset of  $\text{Hil}_n^g(\psi)$  in  $\ker(\Omega_{g,n})$  then  $\hat{\sigma}_1^\psi$  and  $\hat{\sigma}_2^\psi$  are equivalent links in the manifold  $M_\psi$ .*

*Proof.* To prove the equivalence it is enough to exhibit two orientation preserving homeomorphisms  $f : (H_g, \mathcal{A}_n) \longrightarrow (H_g, \mathcal{A}_n)$  and  $\bar{f} : (\bar{H}_g, \bar{\mathcal{A}}_n) \longrightarrow (\bar{H}_g, \bar{\mathcal{A}}_n)$  making the following diagram commute

$$\begin{array}{ccc} (\partial H_g, \partial \mathcal{A}_n) & \xrightarrow{\tau\psi\sigma_1} & (\partial \bar{H}_g, \partial \bar{\mathcal{A}}_n) \\ \downarrow f|_{\partial} & & \downarrow \bar{f}|_{\partial} \\ (\partial H_g, \partial \mathcal{A}_n) & \xrightarrow{\tau\psi\sigma_2} & (\partial \bar{H}_g, \partial \bar{\mathcal{A}}_n) \end{array}$$

In the first case there exists  $\varepsilon \in \text{Hil}_n^g$  such that  $\sigma_2 = \sigma_1\varepsilon$  so we can choose  $f = \varepsilon^{-1}$  and  $\bar{f} = \text{id}$ . In the second case there exists  $\varepsilon \in \text{Hil}_n^g$  such that  $\sigma_2 = \psi^{-1}\varepsilon\psi\sigma_1$  so we can choose  $f = \text{id}$  and  $\bar{f} = \tau\varepsilon\tau^{-1}$ . ■

In the case of classical plat closure Question 1 was solved in [5], where it is shown that two braids determine the same plat closure if and only if they are related by a finite sequence of moves corresponding to generators of  $\text{Hil}_n^0$  and a stabilization move (see also [24]).

Another non trivial question concerns the surjectivity of the map (2). The following Proposition deals with this problem.

PROPOSITION 7 *Let  $M$  be a 3-manifold with a finite number of equivalence classes of genus  $g$  Heegaard splittings<sup>1</sup>. Then there exist  $\psi_1, \dots, \psi_k \in \mathcal{T}_n^g$  such that for each  $(g, n)$ -link  $L \subset M$  we have  $L \cong \hat{\sigma}^{\psi_i}$  with  $\sigma \in \ker(\Omega_{g,n})$  and  $i \in \{1, \dots, k\}$ .*

*Proof.* By result of [4], it is possible to choose elements  $\psi_1, \dots, \psi_k \in \text{MCG}(T_g)$  such that, each  $\psi \in \text{MCG}(T_g)$  that induces an Heegaard decomposition  $H_g \cup_{\psi\tau} \bar{H}_g$  of  $M$ , belongs to the same double coset class of  $\psi_i$  in  $\text{MCG}(T_g)$  modulo  $\mathcal{E}_0^g$ , for a certain

---

<sup>1</sup>Two Heegaard splittings of a manifold  $M$  are said equivalent if there exists an homeomorphism  $f : M \longrightarrow M$  that send the one splitting surface into the other.

$i \in \{1, \dots, k\}$ . Now let  $\psi_1, \dots, \psi_k \in \mathcal{T}_n^g$  such that  $\Omega_{g,n}(\psi_i) = \psi_i$ , for  $i = 1, \dots, k$ . Since  $L$  is a  $(g, n)$ -link in  $M$ , there exists  $\psi \in \text{MCG}_{2n}(\mathbb{T}_g)$  such that  $L = \theta_{g,n}(\psi)$  and  $H_g \cup_{\Omega_{g,n}(\psi)} \bar{H}_g$  is a genus  $g$  Heegaard splitting for  $M$ . So there exist  $\bar{\varepsilon}_1, \bar{\varepsilon}_2 \in \mathcal{E}_0^g$  and  $i \in \{1, \dots, k\}$  such that  $\Omega_{g,n}(\psi_i) = \bar{\varepsilon}_1 \Omega_{g,n}(\psi) \bar{\varepsilon}_2$ . Since  $\Omega_{g,n}$  restricts to a surjective homomorphism  $\mathcal{E}_{2n}^g \longrightarrow \mathcal{E}_0^g$ , then there exists  $\varepsilon_i \in \mathcal{E}_{2n}^g$  such that  $\Omega_{g,n}(\varepsilon_i) = \bar{\varepsilon}_i$ , for  $i = 1, 2$ . If we set  $\sigma = \psi_i^{-1} \varepsilon_1 \psi \varepsilon_2$ , then  $\sigma \in \ker(\Omega_{g,n})$  and  $L \cong \hat{\sigma}^{\psi_i}$ .  $\blacksquare$

The Waldhausen conjecture, which has been proved in [17, 18, 22], tells us that every manifold admits a finite number of homeomorphism classes of irreducible genus  $g$  Heegaard splittings. So, for example, Proposition 7 holds whenever  $g$  is the Heegaard genus of  $M$ . In [9] it is analyzed the case  $g = n = 1$ .

## 5 The Hilden map and the motion groups

In this section we describe the connections between Hilden braid groups and the so-called *motion groups*. We start by recalling few definitions (see [12]).

A *motion* of a compact submanifold  $N$  in a manifold  $M$  is a path  $f_t$  in  $\text{Homeo}_c(M)$  such that  $f_0 = \text{Id}_M$  and  $f_1(N) = N$ , where  $\text{Homeo}_c(M)$  denotes the group of homeomorphisms of  $M$  with compact support. A motion is called *stationary* if  $f_t(N) = N$  for all  $t \in [0, 1]$ . The *motion group*  $\mathcal{M}(M, N)$  of  $N$  in  $M$  is the group of equivalence classes of motion of  $N$  in  $M$  where two motions  $f_t, g_t$  are equivalent if  $(g^{-1}f)_t$  is homotopic relative to endpoints to a stationary motion.

Notice that the motion group of  $k$  points in  $M$  is the braid group  $B_k(M)$ . Moreover, since each motion is equivalent to a motion that fixes a point  $*$  in  $M - N$ , it is possible to define a homomorphism

$$\mathcal{M}(M, N) \longrightarrow \text{Aut}(\pi_1(M - N, *)) \quad (3)$$

sending an element represented by the motion  $f_t$  into the automorphism induced on  $\pi_1(M - N, *)$  by  $f_1$ .

We are mainly interested in the case of links in 3-manifolds. In [12] a finite set of generators for the motion groups  $\mathcal{M}(\mathbb{S}^3, L_n)$  of the  $n$ -component trivial link in  $\mathbb{S}^3$  is given, while a presentation can be found in [1]. Moreover in [13] a presentation for the motion group of all torus links in  $\mathbb{S}^3$  is obtained. On the contrary, there are not known examples of computations of motion groups of links in 3-manifolds different from  $\mathbb{S}^3$ .

In [16] was described how to construct examples of motions of a link  $L$  in  $\mathbb{S}^3$  presented as the plat closure of a braid  $\sigma \in B_{2n}(\mathbb{S}^2)$  using the elements of  $\text{Hil}_n^0 \cap \text{Hil}_n^0(\sigma)$ . In the following Theorem we extend this result to links in 3-manifolds via Hilden braid groups of a surface.



**THEOREM 7** *Let  $\psi \in \mathcal{T}_n^g$  and let  $\hat{\sigma}^\psi$  be a link in  $M_\psi = H_g \cup_{\tau\psi} \bar{H}_g$ , where  $\sigma \in \ker(\Omega_{g,n})$ . There exists a group homomorphism, that we call the Hilden map,  $\mathcal{H}_{\psi\sigma} : \text{Hil}_n^g \cap \text{Hil}_n^g(\psi\sigma) \longrightarrow \mathcal{M}(M_\psi, \hat{\sigma}^\psi)$ , where  $\text{Hil}_n^g(\psi\sigma) = (\psi\sigma)^{-1}\text{Hil}_n^g\psi\sigma$ .*

*Proof.* Let  $\varepsilon$  be a representative of an element in  $\text{Hil}_n^g \cap \text{Hil}_n^g(\psi\sigma)$ . By definition  $\text{Hil}_n^g \subset \ker \Omega_{g,n}$ , so there exists an isotopy  $g : I \times T_g \longrightarrow T_g$  such that  $g(0, \cdot) = g_0 = \text{id}$  and  $g(1, \cdot) = g_1 = \varepsilon$ . Then  $\psi\sigma g(\psi\sigma)^{-1}$  is an isotopy between the identity and  $\psi\sigma\varepsilon(\psi\sigma)^{-1}$ . Moreover by hypothesis the isotopy class of  $\psi\sigma\varepsilon(\psi\sigma)^{-1}$  belongs to  $\text{Hil}_n^g$  and so extends to  $H_g$ . Since the rows of the commutative diagram

$$\begin{array}{ccc} \text{MCG}_n(H_g) & \longrightarrow & \mathcal{E}_{2n}^g \\ \downarrow & & \downarrow \Omega_{g,n} \\ \text{MCG}(H_g) & \longrightarrow & \mathcal{E}_0^g. \end{array} \quad (4)$$

are isomorphisms, there exist two isotopies  $f : I \times H_g \longrightarrow H_g$  and  $\bar{f} : I \times H_g \longrightarrow H_g$  between the identity and an extension of, respectively,  $\varepsilon$  and  $\psi\sigma\varepsilon(\psi\sigma)^{-1}$ . We claim that it is possible to choose  $f$  and  $\bar{f}$  such that they extend, respectively,  $g$  and  $\psi\sigma g(\psi\sigma)^{-1}$ . Indeed if  $g = 0$  we can use the Alexander trick to extend the isotopy from the boundary sphere to the 3-ball. If  $g > 0$  first we extend the isotopy on a system of meridian discs for  $H_g$  not intersecting the system of arcs and then we reduce to the previous case by cutting along them.

We define  $\mathcal{H}_{\psi\sigma}([\varepsilon]) = [F]$  where  $F : I \times M_\psi \longrightarrow M_\psi$  is defined by

$$F(t, x)F_t(x) = \begin{cases} f(t, x) & \text{if } x \in H_g \\ \bar{f}(t, x) & \text{if } x \in \bar{H}_g \end{cases}$$

The commutativity of the following diagram ensures that  $F_t$  is a well-defined homeomorphism of  $M_\psi$

$$\begin{array}{ccc} \partial H_g & \xrightarrow{\tau\psi\sigma} & \partial \bar{H}_g \\ g_t \downarrow & & \downarrow \tau\psi\sigma g_t (\tau\psi\sigma)^{-1} \\ \partial H_g & \xrightarrow{\tau\psi\sigma} & \partial \bar{H}_g. \end{array} \quad (5)$$

It is immediate to check that  $F_0 = \text{id}$  and  $F_1(\hat{\sigma}^\psi) = \hat{\sigma}^\psi$  and so  $F$  is a motion of  $\hat{\sigma}^\psi$  in  $M_\psi$ . Moreover the definition of  $\mathcal{H}_{\psi\sigma}$  does not depend on the homeomorphism choosen as a representative of the element in  $\text{Hil}_n^g \cap \text{Hil}_n^g(\psi\sigma)$ : indeed, if  $\varepsilon'$  is another representative, there exists an isotopy between  $\varepsilon$  and  $\varepsilon'$  fixing  $A_1 \cup \dots \cup A_n$  and so the corresponding motions are equivalent.

In order to prove both that the definition does not depend on the choice of the isotopies and that  $\mathcal{H}_{\psi\sigma}$  is a group homomorphisms we distinguish three cases. For the case of  $g = 0$  we refer to [1, 6, 16]. If  $g > 1$ , the statement follows from the fact

that  $\pi_1(\mathcal{H}(T_g), \text{id}) = 1$ , where  $\mathcal{H}(T_g)$  is the group of orientation preserving homeomorphisms of  $T_g$  (see [15]). If  $g = 1$ , then  $\pi_1(\mathcal{H}(T_g), \text{id}) = \mathbb{Z}$ , see [14]. Nevertheless, since  $\text{MCG}(T_1) \cong \text{MCG}_1(T_1)$ , we can suppose that all the isotopies that we take into consideration fix a point; so the statement follows from  $\pi_1(\mathcal{H}(T_g, P), \text{id}) = 1$  (see [14])

■

In order to use the Hilden map to get informations on motion groups, it is natural to ask if  $\mathcal{H}_{\psi\sigma}$  is surjective and/or injective. Clearly the answer will depend on  $\psi\sigma$ , that is on the ambient manifold and on the considered link. Before giving a (partial) answer in the case of  $\mathbb{S}^3$ , let us recall the main result of [12]

**THEOREM 8 ([12])** *The homomorphism  $(\mathbb{S}^3, L_n) \longrightarrow \text{Aut}(\pi_1(\mathbb{S}^3 - L_n, *)) \cong \mathbb{F}_n$  is injective and  $(\mathbb{S}^3, L_n)$  is generated by:*

$R_i$ : *turn the  $i$ -th circle over, corresponding to the automorphism of  $\mathbb{F}_n$*

$$\rho_i : \begin{cases} x_i \longrightarrow x_i^{-1} \\ x_k \longrightarrow x_k \text{ if } k \neq i \end{cases}$$

$T_j$ : *interchange the  $j$ -th and  $(j+1)$ -th circles, corresponding to the automorphism of  $\mathbb{F}_n$*

$$\tau_j : \begin{cases} x_j \longrightarrow x_{j+1} \\ x_{j+1} \longrightarrow x_j \\ x_h \longrightarrow x_h \text{ if } h \neq j, j+1 \end{cases}$$

$A_{ik}$ : *pull the  $i$ -th circle throught the  $k$ -th circle, corresponding to the automorphism of  $\mathbb{F}_n$*

$$\alpha_{ik} : \begin{cases} x_i \longrightarrow x_k x_i x_k^{-1} \\ x_h \longrightarrow x_h \text{ if } h \neq i \end{cases}$$

where  $j = 1, \dots, n-1$ ,  $i, k = 1, \dots, n$  and  $i \neq k$ .

**COROLLARY 9** *Let  $\psi \in \mathcal{T}_n^g$  be an element such that  $M_\psi = \mathbb{S}^3$ . For example, choose  $\sigma = \text{id}$  if  $g = 0$  and  $\sigma = T_{\alpha_1} \cdots T_{\alpha_g}$ , where  $\alpha_1, \dots, \alpha_g$  denote the curves depicted in Figure 9 if  $g > 1$ . The homomorphism  $\mathcal{H}_\psi : \text{Hil}_n^g \cap \text{Hil}_n^g(\psi) \longrightarrow \mathcal{M}(\mathbb{S}^3, L_n)$  is surjective. Moreover, it is injective if and only if  $(g, n) = (0, 1)$ .*

*Proof.* First of all notice that each element of  $\mathcal{T}_n^g$  commutes with the following elements of  $\text{Hil}_n^g$ :  $\iota_i$ ,  $\lambda_j$ ,  $s_{i,k}$ , for  $j = 1, \dots, n-1$ ,  $i, k = 1, \dots, n$  and  $i \neq k$ . So all these elements belong to  $\text{Hil}_n^g \cap \text{Hil}_n^g(\psi)$ . Moreover,  $\mathcal{H}_\psi(\iota_i) = R_i$ ,  $\mathcal{H}_\psi(\lambda_j) = T_j$  and  $\mathcal{H}_\psi(s_{i,k}) = A_{ik}$  so the surjectivity of  $\mathcal{H}_\psi$  follows by Theorem 8. The same holds for  $\mathcal{M}(\mathbb{S}^3, L_1)$ , since, by Theorem 8, it is isomorphic to the subgroup of  $\text{Aut}(\mathbb{F}_2)$

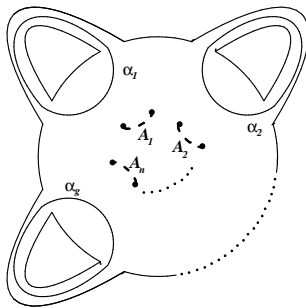


Figure 9: An example of Hilden map.

generated by  $\rho_1$  which has clearly order two. On the contrary, if  $(g, n) \neq (0, 1)$ , then  $\iota_1$  has infinite order in  $\text{MCG}_{2n}(\mathbb{T}_g)$  while  $\rho_1$ , and so  $R_1$ , is an element of order two. ■

Using results from [9] and [13] it would be possible to analyze the case of the torus links in  $\mathbb{S}^3$ . Moreover, the Hilden map could be used in order to get informations on motion groups of links that belongs in 3-manifolds different from  $\mathbb{S}^3$ .

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